A Theory for Enhancing Eruption Prediction through Multivariate Volcanic Data Integration

Minoru Luke Ideno

June 10, 2024

Abstract

This short-paper presents a theoretical framework demonstrating that integrating volcanic activity observation data from multiple perspectives enhances the performance of eruption classification models. By combining individual models' AUC values (AUC_i), the Pearson correlation coefficients among models (Σ), and the number of models (m), we derive an approximate expression for the ensemble model's AUC. This framework provides quantitative insights into the extent of performance improvement, the necessary performance levels and quantities of observation data, and the degree of diversity required among the observational data capturing different aspects of volcanic activity. Additionally, we explore the dependence of Pearson correlation coefficients on the time scales of the observational data, highlighting the benefits of integrating data with diverse temporal dynamics.

1 Introduction

Integrating multiple observational datasets in volcanic monitoring models is generally believed to enable more accurate predictions. However, there is a lack of a theoretical framework to quantitatively address questions such as *how much the performance can be improved*, *how many and what performance levels of observational data are required*, and *to what extent the observational data should capture different aspects of volcanic activity*. This study theoretically analyzes the performance enhancement of ensemble models based on the performance of individual models and the correlations between them. Furthermore, we delve into the relationship between time-scale diversity in observational data and the resulting correlation coefficients, providing deeper insights into the benefits of multivariate data integration.

2 Theoretical Framework

2.1 AUC for Each Observational Data

Each decision tree's output follows a Bernoulli distribution with a fixed probability. Therefore, the sum of multiple decision tree outputs can be represented as a binomial distribution. When the conditions $np > 5$ and $n(1-p) > 5$ are satisfied, the binomial distribution can be approximated by a normal distribution. Specifically, the outputs for positive cases (eruption) and negative cases (non-eruption) follow the normal distributions:

$$S_+ \sim N(\mu_+, \sigma_+^2), \quad S_- \sim N(\mu_-, \sigma_-^2)$$

where S_+ denotes the output for positive cases and S_- for negative cases. The AUC (Area Under the Curve) for each observational dataset, interpreted as the probability that a positive case's output exceeds that of a negative case, is expressed as:

$$\text{AUC}_i = P(S_+^i - S_-^i > 0) = \Phi\left(\frac{\mu_{D_i}}{\sigma_{D_i}}\right)$$

Here, $D_i = S_+^i - S_-^i$ is defined, and since D_i follows a normal distribution $N(\mu_D, \sigma_D^2)$, the AUC can be represented using the cumulative distribution function Φ of the standard normal distribution. The parameters are $\mu_D = \mu_+ - \mu_-$ and $\sigma_D^2 = \sigma_+^2 + \sigma_-^2$.

2.2 AUC for Weighted Ensemble Model

For the ensemble model, the outputs of individual decision trees are scaled by weights w_i and averaged:

$$S_{\text{ensemble}} = \sum_{i=1}^{m} w_i S_i, \quad \text{where } \sum_{i=1}^{m} w_i = 1.$$

The AUC of the ensemble model is then approximated by:

$$\text{AUC}_{\text{ensemble}} = \Phi\left(\frac{\sum_{i=1}^{m} w_i \Phi^{-1}(\text{AUC}_i)}{\sqrt{\boldsymbol{w}^\top \Sigma \boldsymbol{w}}}\right)$$

In this equation, $\boldsymbol{w}$ represents the weight vector, and Σ is the Pearson correlation matrix among the individual models. This expression provides a guideline for selecting optimal weights based on the performance and correlations of individual models to maximize the ensemble model's predictive performance. The theoretical framework supports the expectation that ensemble models integrating multiple diverse observational data sources will exhibit improved prediction performance.

2.3 Derivation of Correlation from Power Spectral Density

Understanding the Pearson correlation coefficients between models is crucial for optimizing the ensemble performance. We derive the correlation coefficients based on the power spectral density of the models' time-series outputs.

2.3.1 Relationship Between Time-Series Data and Power Spectral Density

Let $f_i(t)$ and $f_j(t)$ be the time-series outputs from models i and j, respectively. The power spectral density $P_i(\omega)$ is the Fourier transform of the autocorrelation function $R_i(\tau)$:

$$P_i(\omega) = \int_{-\infty}^{\infty} R_i(\tau) e^{-i\omega\tau} d\tau, \quad R_i(\tau) = \mathbb{E}[f_i(t) f_i(t+\tau)]$$

For two models, the cross-spectral density $P_{ij}(\omega)$ is the Fourier transform of the cross-correlation function:

$$P_{ij}(\omega) = \int_{-\infty}^{\infty} R_{ij}(\tau) e^{-i\omega\tau} d\tau, \quad R_{ij}(\tau) = \mathbb{E}[f_i(t) f_j(t+\tau)]$$

2.3.2 Histogram Representation of Time-Series Data

The histogram S_i approximates the distribution of $f_i(t)$. Utilizing Parseval's theorem, the variance of $f_i(t)$ is related to its spectral density:

$$\mathrm{Var}(f_i) = \int_{-\infty}^{\infty} P_i(\omega) d\omega$$

Similarly, the covariance between $f_i(t)$ and $f_j(t)$ is given by:

$$\mathrm{Cov}(f_i, f_j) = \int_{-\infty}^{\infty} P_{ij}(\omega) d\omega$$

2.3.3 Correlation Coefficient in Terms of Power Spectral Density

The Pearson correlation coefficient ρ_{ij} is defined as:

$$\rho_{ij} = \frac{\mathrm{Cov}(f_i, f_j)}{\sqrt{\mathrm{Var}(f_i)\mathrm{Var}(f_j)}}$$

Substituting the spectral density representations:

$$\rho_{ij} = \frac{\int_{-\infty}^{\infty} P_{ij}(\omega) d\omega}{\sqrt{\int_{-\infty}^{\infty} P_i(\omega) d\omega \cdot \int_{-\infty}^{\infty} P_j(\omega) d\omega}}$$

Assuming $P_{ij}(\omega) \approx P_i(\omega)P_j(\omega)$ for simplicity, the numerator becomes the inner product:

$$\int_{-\infty}^{\infty} P_i(\omega)P_j(\omega) d\omega$$

Thus, the final form of the correlation coefficient is:

$$\rho_{ij} = \frac{\int_{-\infty}^{\infty} P_i(\omega)P_j(\omega) d\omega}{\sqrt{\int_{-\infty}^{\infty} P_i(\omega)^2 d\omega \cdot \int_{-\infty}^{\infty} P_j(\omega)^2 d\omega}}$$

$$\rho_{ij} \propto \int_{-\infty}^{\infty} P_i(\omega)P_j(\omega) d\omega$$

This indicates that the Pearson correlation coefficient is proportional to the inner product of the power spectral densities of the two models. Consequently, observational data capturing different time scales tend to have lower Pearson correlation coefficients, which implies that integrating such diverse data sources can significantly contribute to improving the AUC of the ensemble model.

3 Implications for Volcanic Activity Prediction

The derived theoretical framework provides several key insights for enhancing volcanic activity prediction models:

1. **performance Improvement**: By understanding how individual model performances and their correlations contribute to the ensemble's AUC, practitioners can strategically select and weight models to achieve optimal prediction performance.

2. **Data Requirements**: The framework quantifies the necessary performance levels and the number of observational datasets required to attain desired performance improvements.

3. **Diversity of Observational Data**: Incorporating observational data that capture different aspects or time scales of volcanic activity reduces inter-model correlations, thereby enhancing the ensemble's overall performance.